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1994 J. Phys. A: Math. Gen. 27 3635

(http://iopscience.iop.org/0305-4470/27/11/013)

J. Phys. A: Math. Gen. 27 (1994) 3635–3642. Printed in the UK

Two rigorous theorems on the momentum distribution functions of the Hubbard model at half-filling

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Received 21 February 1994

Abstract. In this article, based on a recent theorem by Lieb *et al*, we shall prove two theorems on the momentum distribution functions of the half-filled Hubbard model on a d-dimensional simple cubic lattice in a mathematically rigorous way. More precisely, we shall first show that the half-filled positive-U and negative-U Hubbard models have the same momentum distribution functions $n_\uparrow(q)$ and $n_\downarrow(q)$. Then, we will show that $n_\sigma(q)$ are symmetric functions about the value $\bar{n} = \frac{1}{2}$. Finally, we shall briefly discuss some possible applications of these theorems to the further numerical investigations on the ground state of the Hubbard model at half-filling.

Attempts to understand the properties of the copper oxide based high-T_c superconductors have led to an increased interest in the models of strongly correlated electrons moving in two spatial dimensions. In particular, the normal-state properties of these materials have brought to attention the inadequacy of the phenomenology of the Fermi-liquid theory. Varma *et al* [1] proposed that the normal state of the copper oxide based superconductors could be described by a marginal Fermi-liquid theory. Noticing the rigorously known breakdown of the Fermi-liquid theory in one dimension [2–6], Anderson [7] suggested that similar non-Fermi-liquid behaviours can also be found in a two-dimensional strongly correlated electron system. A characteristic feature of the non-Fermi liquids is the absence of discontinuity in the momentum distribution function $n(k)$ at the Fermi momentum k_F [3–5]. To determine whether the Hubbard model falls into the Fermi-liquid or non-Fermi-liquid category, many researchers have applied various methods to calculate the momentum distribution function of the Hubbard model [8–14].

In a very recent article, Lieb, Loss and McCann [15] extended a previous result of MacLachlan [16, 17] and proved a very interesting theorem on the one-particle density matrix of the half-filled Hubbard model on a bipartite lattice. In this article, based on the theorem by Lieb *et al*, we shall prove some new rigorous results on the momentum distribution functions of the Hubbard model.

Before proceeding to the statement of our theorems, we would first like to introduce some useful notation and terminologies.

Take a finite d-dimensional simple cubic (SC) lattice Λ with $N_\Lambda = L^d$ lattice points (we let the lattice constant $a = 1$) and impose the periodic boundary condition on Λ. Then, the Hamiltonian of the Hubbard model on Λ can be written as

$$H_\Lambda = -t\sum_\sigma\sum_{\langle ij\rangle}(c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma}) + U\sum_{i\in\Lambda}(n_{i\uparrow} - \tfrac{1}{2})(n_{i\downarrow} - \tfrac{1}{2}) \qquad (1)$$

0305-4470/94/113635+08$19.50

where $c_{i\sigma}^{\dagger}$ ($c_{i\sigma}$) is the electron creation (annihilation) operator which creates (annihilates) an electron of spin σ at lattice site i. $n_{i\sigma} = c_{i\sigma}^{\dagger} c_{i\sigma}$ and $\langle ij \rangle$ denotes a pair of nearest-neighbour lattice points. $t > 0$ and U are two parameters representing the hopping energy and the on-site interaction of electrons, respectively. In the conventional Hubbard Hamiltonian, the parameter U is chosen to be positive for an on-site Coulomb repulsion. However, it is well known [18] that the negative-U Hubbard model also has many applications in condensed matter physics. Therefore, in this article, we shall study both positive- and negative-U Hubbard models. With respect to the Hamiltonian (1), the SC lattice is apparently a bipartite lattice. Namely, it can be divided into two separate sublattices A and B such that electrons can only hop from a site in one sublattice to a site in another sublattice. Furthermore, it is easy to see that the Hamiltonian commutes with the electron-number operator $\hat{N} = \sum_{i,\sigma} c_{i\sigma}^{\dagger} c_{i\sigma}$. Therefore, the number of electrons is a good quantum number. In particular, when the number of electrons is equal to the number of the lattice sites N_{Λ}, the Hubbard model is called half-filled.

With these definitions, the rigorous result of Lieb *et al* [15] can be summarized in the following theorem.

The uniform-density theorem. Let $\Psi_0(\Lambda, U)$ be the ground state of the half-filled Hubbard model on an arbitrary bipartite lattice. For both $U > 0$ and $U < 0$, the elements of the one-particle density matrix of $\Psi_0(\Lambda, U)$ satisfy

$$f_{hs}(\sigma) \equiv \langle \Psi_0 | c_{h\sigma}^{\dagger} c_{s\sigma} | \Psi_0 \rangle = \begin{cases} \frac{1}{2} & \text{if } h = s \\ 0 & \text{if } h \neq s \text{ are in the same sublattice}. \end{cases} \tag{2}$$

Remark 1. In their original article [15], Lieb *et al* proved the uniform-density theorem for a much wider class of strongly correlated electron models. Here, we have tailored their theorem into a simpler form which is more suitable for our following discussions.

Remark 2. The uniform-density theorem tells us a great deal about the reduced one-particle density matrix of the half-filled Hubbard model. However, to study various kinds of long-range orders, such as the antiferromagnetic long-range order in the ground state of this model, one has to investigate the reduced two-particle density matrices (as explained by Yang in [19]).

To begin with, we first order the lattice points of Λ alphabetically and write $\mathcal{F}(\sigma) \equiv (f_{hs}(\sigma))$ in an $N_{\Lambda} \times N_{\Lambda}$ matrix. Since the Hubbard Hamiltonian (1) and its ground state $\Psi_0(\Lambda, U)$ at half-filling are translationally invariant under the periodic boundary condition, the matrix elements of $\mathcal{F}(\sigma)$ satisfy

$$f_{hs}(\sigma) = f(h - s, \sigma). \tag{3}$$

For a matrix subject to condition (3), we have the following lemma.

Lemma 1. Let $A = (a_{ij})$ be an $N_{\Lambda} \times N_{\Lambda}$ matrix defined on a SC lattice Λ. Assume that the elements of A satisfy condition (3). Then, all of its eigenvalues are given by

$$\lambda_q = \frac{1}{N_{\Lambda}} \sum_{i \in \Lambda} \sum_{j \in \Lambda} a_{ij} \exp[iq \cdot (i - j)] \tag{4}$$

where q is a reciprocal vector of Λ.

This lemma is well known in matrix theory. In a previous article [20], this lemma was used to study off-diagonal long-range order [19] in the ground state of the negative-U Hubbard model. For completeness, we give its proof.

Proof. Take an arbitrary reciprocal vector q of Λ. We define vector u_q by

$$u_q(i) \equiv \exp(-\mathrm{i}q \cdot i) \tag{5}$$

where $u_q(i)$ denotes the ith component of u_q. With definition (5), two vectors u_{q_1} and u_{q_2} are perpendicular to each other if $q_1 \neq q_2$. Namely,

$$u_{q_1} \cdot u_{q_2} \equiv \sum_{i \in \Lambda} \overline{u}_{q_1}(i) u_{q_2}(i) = 0. \tag{6}$$

We now show that these vectors are actually the eigenvectors of A.

By the definitions of matrix A and vector u_q, we have

$$\begin{aligned} (Au_q)(i) &= \sum_{j \in \Lambda} a_{ij} u_q(j) = \sum_{j \in \Lambda} a(i-j) \exp(-\mathrm{i}q \cdot j) \\ &= \exp(-\mathrm{i}q \cdot i) \sum_{j \in \Lambda} a(i-j) \exp[\mathrm{i}q \cdot (i-j)] \\ &= \exp(-\mathrm{i}q \cdot i) \left\{ \frac{1}{N_\Lambda} \sum_{i \in \Lambda} \sum_{j \in \Lambda} a(i-j) \exp[\mathrm{i}q \cdot (i-j)] \right\}. \end{aligned} \tag{7}$$

Therefore, u_q is an eigenvector of A corresponding to the eigenvalue λ_q given in (4). On the other hand, the total number of reciprocal vectors of Λ is equal to N_Λ. Consequently, $\{\lambda_q\}$ is the complete set of the eigenvalues of A which is an $N_\Lambda \times N_\Lambda$ matrix. □

Applying this lemma to the one-particle density matrix $\mathcal{F}(\sigma)$ of $\Psi_0(\Lambda, U)$, we find that the eigenvalues of $\mathcal{F}(\sigma)$ are

$$\lambda_q(\sigma) = \frac{1}{N_\Lambda} \sum_{h \in \Lambda} \sum_{s \in \Lambda} f_{hs}(\sigma) \exp[\mathrm{i}q \cdot (h-s)] = \langle \Psi_0(\Lambda, U) | c_{q\sigma}^\dagger c_{q\sigma} | \Psi_0(\Lambda, U) \rangle \equiv n_\sigma(q) \tag{8}$$

where $c_{q\sigma} = 1/\sqrt{N_\Lambda} \sum_{i \in \Lambda} c_{i\sigma} \exp(-\mathrm{i}q \cdot i)$, and q is a reciprocal vector. In other words, the totality of the eigenvalues of $\mathcal{F}(\sigma)$ is actually the momentum distribution function of electrons of spin σ. Therefore, our knowledge of the one-particle density matrix $\mathcal{F}(\sigma)$ of $\Psi_0(\Lambda, U)$ can be used to derive the properties of the momentum distribution function of the translationally invariant ground state. In particular, we would expect that the uniform-density theorem tells us a great deal about the momentum distribution functions of the half-filled Hubbard Hamiltonian on a d-dimensional SC lattice.

First, as an application of the uniform-density theorem, we prove

Theorem 1. The momentum distribution functions of both half-filled positive-U and negative-U Hubbard Hamiltonians on the same SC lattice are identical.

Proof. As concluded above, to prove theorem 1 we need only show that the one-particle matrices of both positive-U and negative-U Hubbard Hamiltonians at half-filling have the same eigenvalues. For this purpose, we exploit the well known unitary partial particle–hole transformation [21]

$$\hat{U}_0 c_{i\uparrow} \hat{U}_0^\dagger = c_{i\uparrow} \qquad \hat{U}_0 c_{i\downarrow} \hat{U}_0^\dagger = \exp(-i\pi \cdot i) c_{i\downarrow}^\dagger \tag{9}$$

where $\pi = (\pi, \ldots, \pi)$. Under this transformation, the positive-U Hubbard Hamiltonian (1) is transformed into the negative-U Hubbard Hamiltonian of the same form. Since the transformation is unitary, both Hamiltonians should have the same energy spectrum. In particular, the global ground state of the positive-U Hubbard Hamiltonian is mapped onto its negative-U counterpart. On the other hand, it has been shown that the global ground state of the Hubbard Hamiltonian (1) on an arbitrary bipartite lattice coincides with its ground state $\Psi_0(\Lambda, U)$ at half-filling†. Therefore, under the partial particle–hole transformation, $\Psi_0(\Lambda, U)$ is mapped onto $\Psi_0(\Lambda, -U)$. Consequently, by using the first equality of (9), we immediately obtain

$$\langle \Psi_0(\Lambda, U) | c_{h\uparrow}^\dagger c_{s\uparrow} | \Psi_0(\Lambda, U) \rangle = \langle \Psi_0(\Lambda, -U) | c_{h\uparrow}^\dagger c_{s\uparrow} | \Psi_0(\Lambda, -U) \rangle. \tag{10}$$

Namely, $\mathcal{F}(\uparrow, U) = \mathcal{F}(\uparrow, -U)$ at half-filling and hence,

$$n_\uparrow(q, U) = n_\uparrow(q, -U). \tag{11}$$

Next, we show that $\mathcal{F}(\downarrow, U) = \mathcal{F}(\downarrow, -U)$ as well as equation (11).

We notice that, by the uniform-density theorem, we need only consider those matrix elements $f_{hs}(\downarrow)$ with h and s in different sublattices. Under the partial particle–hole transformation

$$\begin{aligned} f_{hs}(\downarrow, U) &\equiv \langle \Psi_0(\Lambda, U) | c_{h\downarrow}^\dagger c_{s\downarrow} | \Psi_0(\Lambda, U) \rangle \\ &= \langle \Psi_0(\Lambda, U) | \hat{U}_0^\dagger (\hat{U}_0 c_{h\downarrow}^\dagger \hat{U}_0^\dagger)(\hat{U}_0 c_{s\downarrow} \hat{U}_0^\dagger) \hat{U}_0 | \Psi_0(\Lambda, U) \rangle \\ &= \langle \Psi_0(\Lambda, -U) | \exp(\mathrm{i}\pi \cdot h) c_{h\downarrow} \exp(-\mathrm{i}\pi \cdot s) c_{s\downarrow}^\dagger | \Psi_0(\Lambda, -U) \rangle \\ &= (-1) \exp[\mathrm{i}\pi \cdot (h - s)] \langle \Psi_0(\Lambda, -U) | c_{s\downarrow}^\dagger c_{h\downarrow} | \Psi_0(\Lambda, -U) \rangle \equiv f_{sh}(\downarrow, -U). \end{aligned} \tag{12}$$

(In the last step of the derivation of (12), we used the fact that h and s are in different sublattices and hence, $\exp[\mathrm{i}\pi \cdot (h - s)] = -1$.) Therefore, $\mathcal{F}(\downarrow, U) = \mathcal{F}^{\mathrm{T}}(\downarrow, -U)$, the transpose of the matrix $\mathcal{F}(\downarrow, -U)$. On the other hand, since the Hubbard Hamiltonian (1) on the finite lattice Λ is a *real* matrix, we can write its global ground-state wavefunction $\Psi_0(\Lambda, -U)$ as a *real* linear combination of a *real* basis of vectors. Consequently, all the matrix elements of $\mathcal{F}(\downarrow, -U)$ are *real* numbers and hence, we have

$$\mathcal{F}(\downarrow, U) = \mathcal{F}^{\mathrm{T}}(\downarrow, -U) = \mathcal{F}^\dagger(\downarrow, -U) = \mathcal{F}(\downarrow, -U). \tag{13}$$

Therefore, the one-particle density matrices $\mathcal{F}(\downarrow, U)$ and $\mathcal{F}(\downarrow, -U)$ have the same eigenvalues and hence identity (11) also holds for the momentum distribution functions of down-spin electrons.

Our proof is accomplished. □

† This fact is well known for the Hubbard Hamiltonian (1) in the thermodynamic limit. For a finite bipartite lattice Λ, it was recently proved by E H Lieb. We thank Professor Lieb for showing us his results before publication.

Some remarks are in order.

Remark 3. In one dimension, the conclusion of theorem 1 is well known. In a seminal article, Mattis and Lieb [4] rigorously solved the one-dimensional Luttinger model [3] which represents a strongly correlated spinless fermion system. By using this exact solution, they showed explicitly that the momentum distribution function of this system is independent of the sign of interactions.

Remark 4. Theorem 1 has some very interesting physical implications. Originally, the negative-U Hubbard model was introduced to describe the real-space-pairing superconductors [18] while its positive-U counterpart is mainly used to study the metal–insulator transition in a strongly correlated electron system [22]. When $|U|$ is sufficiently large, one would expect that the ground state of the negative-U Hubbard model is a Bose–Einstein condensate of the paired electrons, which behave like bosons. For such a system, the Fermi surface in the reciprocal vector space is meaningless. On the other hand, theorem 1 tells us that the momentum distribution functions of both negative-U and positive-U Hubbard models at half-filling are identical. Therefore, the Fermi surface should also be absent in the ground state of the positive-U Hubbard model at half-filling. In other words, the half-filled positive-U Hubbard model describes a non-Fermi liquid. In one dimension, this fact is obvious. By the exact solution of Lieb and Wu [23], the ground state of the positive-U Hubbard model at half-filling is insulating for any $U > 0$. Therefore, it cannot be a Fermi liquid. In $d \geqslant 2$ dimensions, the situation is rather complicated. When $U > 2dt$, the ground state of the half-filled Hubbard model is apparently insulating. However, when $0 < U < 2dt$, there is no rigorous result known. By using some approximate methods, such as mean-field theory and Monte Carlo calculations [24], we found that, in this case, the ground state of the half-filled Hubbard model has a spin-wave energy gap which renders the system an insulator. Consequently, the ground state is still not a Fermi liquid. Our rigorous theorem is consistent with this picture.

Remark 5. Obviously, the identity $\mathcal{F}(\downarrow, U) = \mathcal{F}(\downarrow, -U)$ can be shown directly by using the unitary partial particle–hole transformation for down spins. To avoid introducing an unnecessary new transformation, we applied the uniform-density theorem to prove the identity.

Next, we proceed to show another corollary of the uniform-density theorem. It gives us more detailed information on the momentum distribution functions of the Hubbard model at half-filling.

Theorem 2. The momentum distribution functions of the half-filled Hubbard model on a d-dimensional SC lattice are symmetric about value $\bar{n} = \frac{1}{2}$. In other words, if $n_\sigma(q_1) = \frac{1}{2} + \delta$ ($\delta \geqslant 0$ is a constant) for some reciprocal vector q_1, then there must be another reciprocal vector q_2 at which $n_\sigma(q_2) = \frac{1}{2} - \delta$.

Proof. To prove this theorem, we exploit the bipartite property of the d-dimensional SC lattices with respect to the Hubbard Hamiltonian (1).

By properly grouping the indices of the matrix elements of $\mathcal{F}(\sigma)$, we can rewrite it into a new matrix $\widetilde{\mathcal{F}}(\sigma)$ with the following block form

$$\widetilde{\mathcal{F}}(\sigma) = \begin{pmatrix} \widetilde{\mathcal{F}}_{AA}(\sigma) & \widetilde{\mathcal{F}}_{AB}(\sigma) \\ \widetilde{\mathcal{F}}_{BA}(\sigma) & \widetilde{\mathcal{F}}_{BB}(\sigma) \end{pmatrix} \tag{14}$$

where each block is an $N_\Lambda/2 \times N_\Lambda/2$ submatrix. By the uniform-density theorem, we can easily determine $\widetilde{\mathcal{F}}_{AA}(\sigma)$ and $\widetilde{\mathcal{F}}_{AA}(\sigma)$.

$$\widetilde{\mathcal{F}}_{AA}(\sigma) = \widetilde{\mathcal{F}}_{BB}(\sigma) = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} \end{pmatrix}. \tag{15}$$

Obviously, $\widetilde{\mathcal{F}}(\sigma)$ is unitarily equivalent to $\mathcal{F}(\sigma)$. Therefore, they have the same eigenvalues.

From elementary linear algebra, we know that the eigenvalues of $\widetilde{\mathcal{F}}(\sigma)$ are given by the solutions of its characteristic equation

$$\det(\lambda I - \widetilde{\mathcal{F}}(\sigma)) = 0. \tag{16}$$

On the other hand, for a matrix in block form (14), calculation of its determinant can be made easier by using the following lemma.

Lemma 2. Let M be a $2N \times 2N$ matrix of the following form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{17}$$

where A, B, C and D are $N \times N$ square submatrices. For such a matrix, we have

$$\det M = \det A \det(D - CA^{-1}B). \tag{18}$$

In particular, if A commutes with C then $\det M = \det(AD - CB)$ and it holds even if A has no inverse. A proof of this lemma can be found on page 17 of [25].

Applying this lemma to matrix $\widetilde{\mathcal{F}}(\sigma)$, we immediately obtain

$$\begin{aligned}\det(\lambda I - \widetilde{\mathcal{F}}(\sigma)) &= \det[(\lambda - \tfrac{1}{2})^2 I - \widetilde{\mathcal{F}}_{AB}(\sigma)\widetilde{\mathcal{F}}_{BA}(\sigma)] \\ &= \det[(\lambda - \tfrac{1}{2})^2 I - \widetilde{\mathcal{F}}_{AB}(\sigma)\widetilde{\mathcal{F}}^{\dagger}_{AB}(\sigma)] = 0\end{aligned} \tag{19}$$

since $(\lambda - \frac{1}{2})I$ commutes with any matrix. Therefore, if $\lambda_1 = \frac{1}{2} + \delta$ is a root of the characteristic equation of $\widetilde{\mathcal{F}}(\sigma)$, $\lambda_2 = \frac{1}{2} - \delta$ must also be a root, i.e. the eigenvalues of $\widetilde{\mathcal{F}}(\sigma)$ are symmetrically paired about $\bar{n} = \frac{1}{2}$. Consequently, the momentum distribution functions $n_\sigma(q)$ are symmetric about $\bar{n} = \frac{1}{2}$.

Theorem 2 is proved. □

Remark 6. It is interesting to see that, when the interaction between electrons is absent, theorem 2 is a direct consequence of the Pauli exclusion principle. In fact, with $U = 0$, the Hubbard Hamiltonian is reduced to

$$H_0 = -t \sum_\sigma \sum_q (\cos q_1 + \cdots + \cos q_d) c^{\dagger}_{q\sigma} c_{q\sigma} \tag{20}$$

after a Fourier transformation. According to the Pauli exclusion principle, when temperature $T = 0$, the lowest energy levels will be first filled up by electrons and there is exactly one electron of spin σ per each occupied energy level. Therefore, the momentum distribution functions of this system are of the following form

$$n_\sigma(q) = \begin{cases} 1 & \text{if } |q| \leqslant |k_{\mathrm{F}}| \\ 0 & \text{if } |q| > |k_{\mathrm{F}}| \end{cases} \tag{21}$$

where k_{F} is the Fermi momentum. In particular, when the lattice is half-filled, $n_\sigma(q)$ is apparently symmetric about $\bar{n} = \frac{1}{2}$.

Notice that, if the ground state of a strongly correlated electron system is a non-Fermi liquid then its momentum distribution functions $n_\sigma(q)$ should be continuous at the Fermi surface. Consequently, $n_\sigma(q)$ are continuous functions in the reciprocal vector space. Therefore, a direct corollary of theorem 2 is:

Corollary of theorem 2. A necessary condition for the ground state of the half-filled Hubbard model on a SC lattice to be a non-Fermi liquid is, in the thermodynamic limit,

$$n_\sigma(q) = \tfrac{1}{2} \tag{22}$$

which holds for at least one reciprocal vector $\boldsymbol{q}$.

Obviously, theorem 2 and its corollary alone are not sufficient to determine whether there is a discontinuity of $n_\sigma(q)$ on the Fermi surface. However, under some circumstances, theorem 2 becomes more restrictive and may be useful for further numerical investigation of the properties of the Hubbard models. For examples, we shall consider the following cases.

Case 1: The one-dimensional Hubbard model. The momentum distribution functions of the one-dimensional Hubbard model at half-filling have been studied intensively by many researchers [8–10]. Many physicists believe that the momentum distribution functions of the ground state $\Psi_0(\Lambda, U)$ is a non-increasing function of q and has the following form

$$n_\sigma(q) = a - b\,\mathrm{sgn}(q - k_F)|q - k_F|^\theta \tag{23}$$

when q is near k_F. In formula (23), a, b and θ are constants. If this is true, then, by theorem 2, we must have:

(i) the constant a in formula (23) equal to $\bar{n} = \frac{1}{2}$; and

(ii) if

$$\widetilde{n}_\sigma(q) = \begin{cases} 1 - n_\sigma(q) & \text{for } 0 \leqslant q \leqslant k_F = \pi \\ n_\sigma(q) & \text{for } k_F < q \leqslant 2\pi \end{cases} \tag{24}$$

then, $\widetilde{n}_\sigma(q)$ are symmetric functions about the vertical line $q = k_F = \pi$.

Case 2: The half-filled Hubbard models on the d-dimensional SC lattices with $d \geqslant 2$. In this case, we conjecture that the momentum distribution functions $n_\sigma(q)$ are also non-increasing functions of $\epsilon(q)$. Namely, if

$$\begin{aligned}\epsilon(q_1) &\equiv -t(\cos q_{11} + \cos q_{12} + \cdots + \cos q_{1d}) \\ &\leqslant -t(\cos q_{21} + \cos q_{22} + \cdots + \cos q_{2d}) \equiv \epsilon(q_2)\end{aligned} \tag{25}$$

then $n_\sigma(q_1) \geqslant n_\sigma(q_2)$. When $U = 0$, this conjecture is apparently true. It is also supported by recent numerical calculations [13, 14]. If this conjecture holds, then, by using theorem 2, we find that the ground state $\Psi_0(\Lambda, U)$ of the half-filled Hubbard model on a d-dimensional SC lattice is a non-Fermi liquid if, and only if, $n_\sigma(k_F) = \frac{1}{2}$. Further study on this conjecture is continuing.

Before we finish this article, we would like to make two further remarks.

Remark 6. As we have shown above, the proofs of theorem 1 and 2 are completely based on the uniform-density theorem. In their original paper [15], Lieb *et al* actually proved this theorem for both the $T = 0$ and $T \neq 0$ cases. Therefore, theorems 1 and 2 can be easily extended to the cases of non-zero temperature.

Remark 7. In the proof of the uniform-density theorem, Lieb *et al* mainly used the unitary particle–hole transformation introduced by MacLachlan [16, 17]. Similarly, in our proof of theorem 1, we exploited the well known unitary partial particle–hole transformation [21] between the positive-U and negative-U Hubbard Hamiltonians. Therefore, if the Hamiltonians of some strongly correlated electron systems are also invariant under these unitary transformations then the theorems of this article should also be applicable to these systems.

Acknowledgments

We would like to thank Professor E H Lieb for correspondence. We would also like to thank our referees for their valuable and constructive suggestions. This work was partially supported by the Chinese National Science Foundation under grant no. 19274003.

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